Electrostatic Field due to an Arbitrary Charge Distribution

EE 141 Lecture Notes
Topic 10

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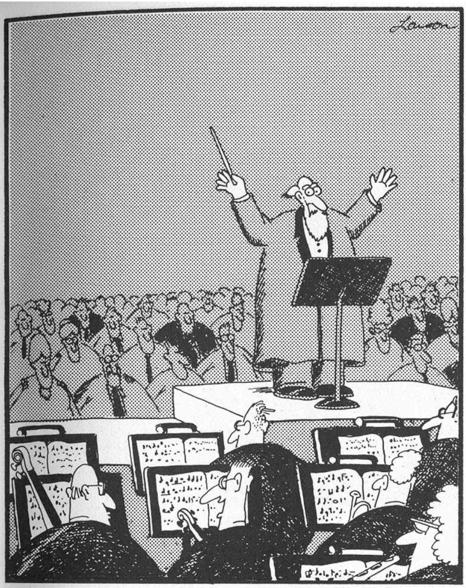
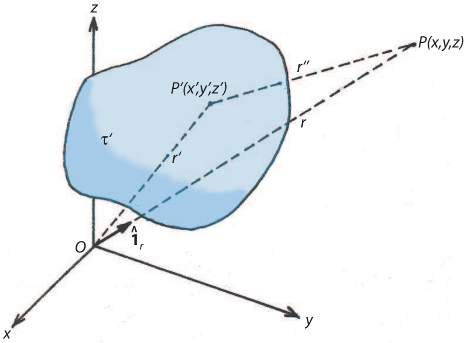
Motivation



"Gee . . . look at all the little black dots."

E - Field Outside an Arbitrary Charge Distribution

Consider an arbitrary (but fixed) charge distribution with density $\varrho(\mathbf{r}')$ occupying a region τ' in free-space and extending to a maximum distance r'_{max} from the origin O of a coordinate system.



It is assumed that the origin O is positioned either within the charged region τ' or is within close proximity to it.

E - Field Outside an Arbitrary Charge Distribution

The absolute electrostatic potential $V(\mathbf{r}) = V(x, y, z)$ at some fixed observation point $P(x, y, z)$ at a distance $r > r'_{max}$ from O is given by

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \iiint_{\tau'} \frac{\varrho(x', y', z')}{r''} d^3r' \quad (1)$$

where

$$r'' = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2} \quad (2)$$

is the distance from the elemental source point at $P'(x', y', z')$ to the field point at $P(x, y, z)$.

Because the source point $P'(x', y', z')$ is taken to be near to the origin O whereas the field point $P(x, y, z)$ is taken to be removed from the origin, one may then expand the quantity $1/r''$ in a Taylor series about the origin $(x', y', z') = (0, 0, 0)$ as follows:

E - Field Outside an Arbitrary Charge Distribution

$$\frac{1}{r''} = \left.\frac{1}{r''}\right|_O + \left(x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right)\left.\frac{1}{r''}\right|_O + \frac{1}{2!}\left(x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right)^2 \left.\frac{1}{r''}\right|_O + \cdots, \quad (3)$$

where $O \implies (x', y', z') = (0, 0, 0)$, and

$$\left(x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right)^2 = x'^2\frac{\partial^2}{\partial x'^2} + y'^2\frac{\partial^2}{\partial y'^2} + z'^2\frac{\partial^2}{\partial z'^2} + 2x'y'\frac{\partial^2}{\partial x'\partial y'} + 2x'z'\frac{\partial^2}{\partial x'\partial z'} + 2y'z'\frac{\partial^2}{\partial y'\partial z'}.$$

E - Field Outside an Arbitrary Charge Distribution

The first term in the Taylor series expansion (3) is given by

$$\left.\frac{1}{r''}\right|_O = \left.\frac{1}{r''}\right|_{(x',y',z')=(0,0,0)} = \frac{1}{r}.$$

For the second term in the Taylor series expansion (3), one has that

$$\begin{aligned}
\left.\frac{\partial}{\partial x'}\frac{1}{r''}\right|_O &= \left.-\frac{1}{r''^2}\frac{\partial r''}{\partial x'}\right|_O = \left.\frac{x-x'}{r''^3}\right|_O = \frac{x}{r^3} = \frac{\ell}{r^2}, \\
\left.\frac{\partial}{\partial y'}\frac{1}{r''}\right|_O &= \frac{y}{r^3} = \frac{m}{r^2}, \\
\left.\frac{\partial}{\partial z'}\frac{1}{r''}\right|_O &= \frac{z}{r^3} = \frac{n}{r^2},
\end{aligned}$$

where $\ell \equiv x/r$, $m \equiv y/r$, and $n \equiv z/r$ are the cosines of the angles between the position vector $\mathbf{r}$ and the x, y, and z-axes, respectively.

E - Field Outside an Arbitrary Charge Distribution

For the third term in the Taylor series expansion (3), one has that

$$
\begin{aligned}
\frac{\partial^2}{\partial x'^2}\frac{1}{r''}\bigg|_O &= \frac{\partial}{\partial x'}\frac{x-x'}{r''^3}\bigg|_O = \frac{-r''^3 - 3(x-x')r''^2\partial r''/\partial x'}{r''^6}\bigg|_O \\
&= \frac{3(x^2/r^2)-1}{r^3} = \frac{3\ell^2-1}{r^3}, \\
\frac{\partial^2}{\partial y'^2}\frac{1}{r''}\bigg|_O &= \frac{3m^2-1}{r^3}, \\
\frac{\partial^2}{\partial z'^2}\frac{1}{r''}\bigg|_O &= \frac{3n^2-1}{r^3},
\end{aligned}
$$

and

E - Field Outside an Arbitrary Charge Distribution

$$\left.\frac{\partial^2}{\partial x'\partial y'}\frac{1}{r''}\right|_O = \left.\frac{\partial^2}{\partial y'\partial x'}\frac{1}{r''}\right|_O = \left.\frac{\partial}{\partial y'}\frac{x-x'}{r''^3}\right|_O = \left.-3\frac{(x-x')}{r''^4}\frac{\partial r''}{\partial y'}\right|_O$$

$$= \left.3\frac{(x-x')(y-y')}{r''^5}\right|_O = 3\frac{xy}{r^5} = 3\frac{\ell m}{r^3},$$

$$\left.\frac{\partial^2}{\partial x'\partial z'}\frac{1}{r''}\right|_O = \left.\frac{\partial^2}{\partial z'\partial x'}\frac{1}{r''}\right|_O = 3\frac{\ell n}{r^3},$$

$$\left.\frac{\partial^2}{\partial y'\partial z'}\frac{1}{r''}\right|_O = \left.\frac{\partial^2}{\partial z'\partial y'}\frac{1}{r''}\right|_O = 3\frac{mn}{r^3}.$$

E - Field Outside an Arbitrary Charge Distribution

With the Taylor series expansion (3) of $1/r''$ about the origin O, the electrostatic potential (1) at the field point $P(x, y, z)$ becomes

$$\begin{aligned}
V(x,y,z) &= \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{r}\iiint_{\tau'}\varrho(x',y',z')d^3r'\right. \\
&\quad +\frac{1}{r^2}\iiint_{\tau'}(\ell x' + my' + nz')\varrho(x',y',z')d^3r' \\
&\quad +\frac{1}{r^3}\iiint_{\tau'}\Big[3\ell m x'y' + 3\ell n x'z' + 3mny'z' \\
&\qquad +\frac{1}{2}(3\ell^2-1)x'^2 + \frac{1}{2}(3m^2-1)y'^2 \\
&\qquad \left. +\frac{1}{2}(3n^2-1)z'^2\Big]\varrho(x',y',z')d^3r' + \cdots\right\} \\
&= V_0(x,y,z) + V_1(x,y,z) + V_2(x,y,z) + \cdots \, . \qquad (4)
\end{aligned}$$

E - Field Outside an Arbitrary Charge Distribution: The Monopole Term $V_0(\mathbf{r})$

The first (or zeroth-order) term in this expansion is called the monopole term because it is simply the electrostatic potential one would have at P if the entire charge distribution were concentrated at the origin O, where

$$V_0(x, y, z) = \frac{1}{4\pi\epsilon_0 r} \underbrace{\iiint_{\tau'} \varrho(x', y', z') d^3 r'}_{\text{net charge } Q \text{ in } \tau'} = \frac{Q}{4\pi\epsilon_0 r}. \tag{5}$$

The monopole term is zero only if the total net charge Q in the region τ' is zero. If it is nonvanishing, then it is the dominant term in the series as $r \to \infty$ because it decreases only as r^{-1}.

E - Field Outside an Arbitrary Charge Distribution: The Dipole Term $V_1(\mathbf{r})$

The second (or first-order) term in the expansion (4) is called the dipole term and may be written as

$$V_1(x, y, z) = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}, \tag{6}$$

$$\hat{\mathbf{r}} \equiv \hat{\mathbf{1}}_x \ell + \hat{\mathbf{1}}_y m + \hat{\mathbf{1}}_z n \tag{7}$$

denoting the unit vector along the radial line from the origin O to the observation (or field) point $P(x, y, z)$, where

$$\mathbf{p} \equiv \iiint_{\tau'} \underbrace{\left(\hat{\mathbf{1}}_x x' + \hat{\mathbf{1}}_y y' + \hat{\mathbf{1}}_z z'\right)}_{\mathbf{r}'} \varrho(x', y', z') d^3 r' = \iiint_{\tau'} \mathbf{r}' \varrho(\mathbf{r}') d^3 r' \tag{8}$$

is the dipole moment of the charge distribution with respect to O. Hence, $V_1(\mathbf{r}) = V_1(x, y, z)$ is the electrostatic potential at $P(x, y, z)$ due to an effective dipole at the origin O with dipole moment $\mathbf{p}$.

E - Field Outside an Arbitrary Charge Distribution: The Dipole Term $V_1(\mathbf{r})$

The dipole moment of an extended charge distribution can also be defined as

$$\mathbf{p} = Q\bar{\mathbf{r}}', \tag{9}$$

where Q is the total net charge in the distribution, as given in Eq.(5), and where $\bar{\mathbf{r}}'$ is a vector extending from the origin O to the charge centroid of the distribution, given by

$$\bar{\mathbf{r}}' = \frac{\iiint_{\tau'} \mathbf{r}'\varrho(\mathbf{r}')d^3r'}{\iiint_{\tau'} \varrho(\mathbf{r}')d^3r'} = \frac{1}{Q}\iiint_{\tau'} \mathbf{r}'\varrho(\mathbf{r}')d^3r'. \tag{10}$$

Notice that if $Q = 0$, then $\bar{\mathbf{r}}' \to \infty$ and $\mathbf{p}$, as given by Eq. (9), is indeterminate; however, Eq. (8) always determines the dipole moment unambiguously. When $Q = 0$, the dipole moment is independent of the choice of origin. If $Q \neq 0$, the dipole moment of the charge distribution can always be made to vanish by choosing the origin O at the centroid of the charge distribution ($\bar{\mathbf{r}}' = \mathbf{0}$).

E - Field Outside an Arbitrary Charge Distribution: The Quadrupole Term $V_2(\mathbf{r})$

The third (or second-order) term in the expansion (4) is called the quadrupole term and may be written as

$$\begin{aligned} V_3(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0 r^3}\Bigg[3\ell m \iiint_{\tau'} x'y'\varrho(\mathbf{r}')d^3r' + 3\ell n \iiint_{\tau'} x'z'\varrho(\mathbf{r}')d^3r' \\ &\quad +3mn \iiint_{\tau'} y'z'\varrho(\mathbf{r}')d^3r' \\ &\quad +\frac{1}{2}(3\ell^2-1)\iiint_{\tau'} x'^2\varrho(\mathbf{r}')d^3r' \\ &\quad +\frac{1}{2}(3m^2-1)\iiint_{\tau'} y'^2\varrho(\mathbf{r}')d^3r' \\ &\quad +\frac{1}{2}(3n^2-1)\iiint_{\tau'} z'^2\varrho(\mathbf{r}')d^3r'\Bigg]. \qquad (11) \end{aligned}$$

E - Field Outside an Arbitrary Charge Distribution: The Quadrupole Term $V_2(\mathbf{r})$

These six integrals define the quadrupole moment of the charge distribution, where

$$\mathcal{Q}_{xx} \equiv \iiint_{\tau'} x'^2 \varrho(\mathbf{r}') d^3 r' = Q\overline{x'^2}, \tag{12}$$

$$\mathcal{Q}_{yy} \equiv \iiint_{\tau'} y'^2 \varrho(\mathbf{r}') d^3 r' = Q\overline{y'^2}, \tag{13}$$

$$\mathcal{Q}_{zz} \equiv \iiint_{\tau'} z'^2 \varrho(\mathbf{r}') d^3 r' = Q\overline{z'^2}, \tag{14}$$

$$\mathcal{Q}_{xy} = \mathcal{Q}_{yx} \equiv \iiint_{\tau'} x'y' \varrho(\mathbf{r}') d^3 r' = Q\overline{x'y'}, \tag{15}$$

$$\mathcal{Q}_{xz} = \mathcal{Q}_{zx} \equiv \iiint_{\tau'} x'z' \varrho(\mathbf{r}') d^3 r' = Q\overline{x'z'}, \tag{16}$$

$$\mathcal{Q}_{yz} = \mathcal{Q}_{zy} \equiv \iiint_{\tau'} y'z' \varrho(\mathbf{r}') d^3 r' = Q\overline{y'z'}. \tag{17}$$

E - Field Outside an Arbitrary Charge Distribution: The Quadrupole Term $V_2(\mathbf{r})$

With these results, Eq. (11) becomes

$$V_3(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\Big[3\ell m\mathcal{Q}_{xy} + 3\ell n\mathcal{Q}_{xz} + 3mn\mathcal{Q}_{yz} + \frac{1}{2}(3\ell^2 - 1)\mathcal{Q}_{xx} + \frac{1}{2}(3m^2 - 1)\mathcal{Q}_{yy} + \frac{1}{2}(3n^2 - 1)\mathcal{Q}_{zz}\Big] \tag{18}$$

E - Field Outside an Arbitrary Charge Distribution: Quadrupole Term $V_2(\mathbf{r})$ - Cylindrical Symmetry

If the charge distribution in τ' possesses cylindrical symmetry about the z-axis, for example, then

$$\mathcal{Q}_{xy} = \mathcal{Q}_{yz} = \mathcal{Q}_{xz} = 0,$$

$$\mathcal{Q}_{xx} = \mathcal{Q}_{yy}.$$

It is then convenient to define a single (scalar) quadrupole moment $\mathcal{Q}$ of the charge distribution as

$$\mathcal{Q} \equiv 2(\mathcal{Q}_{zz} - \mathcal{Q}_{xx}) = \iiint_{\tau'} (3z'^2 - r'^2)\varrho(\mathbf{r}')d^3r', \qquad (19)$$

where $r'^2 = x'^2 + y'^2 + z'^2$.

E - Field Outside an Arbitrary Charge Distribution: Quadrupole Term $V_2(\mathbf{r})$ - Cylindrical Symmetry

Under these conditions and with this definition, Eq. (18) becomes

$$
\begin{aligned}
V_3(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0 r^3}\left[\frac{1}{2}(3\ell^2+3m^2-2)\mathcal{Q}_{xx}+\frac{1}{2}(3n^2-1)\underbrace{\mathcal{Q}_{zz}}_{\frac{1}{2}\mathcal{Q}+\mathcal{Q}_{xx}}\right] \\
&= \frac{1}{4\pi\epsilon_0 r^3}\left[\frac{3}{2}\Big(\underbrace{(\ell^2+m^2+n^2)}_{1}-1\Big)\mathcal{Q}_{xx}+\frac{1}{4}(3n^2-1)\mathcal{Q}\right] \\
&= \frac{\mathcal{Q}}{4\pi\epsilon_0}\frac{3n^2-1}{4r^3}=\frac{\mathcal{Q}}{4\pi\epsilon_0}\frac{3\cos^2\theta-1}{4r^3} \qquad (20)
\end{aligned}
$$

where $n \equiv z/r = \cos\theta$.

E - Field Outside an Arbitrary Charge Distribution: The Multipole Expansion of the Potential

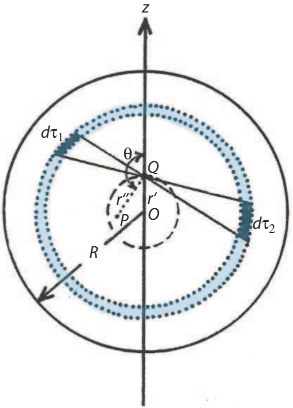
The multipole expansion (4) of the electrostatic potential (1) at the field point $P(x, y, z)$ due to the charge distribution $\varrho(\mathbf{r}')$ in the region τ' about the origin O becomes

$$\begin{aligned} V(x,y,z) &= V_0(x,y,z) + V_1(x,y,z) + V_2(x,y,z) + \cdots \\ &= \underbrace{\frac{Q}{4\pi\epsilon_0 r}}_{\text{monopole term}} + \underbrace{\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}}_{\text{dipole term}} + \underbrace{\frac{\mathcal{Q}}{4\pi\epsilon_0 r^3}\frac{3\cos^2\theta - 1}{4}}_{\text{quadrupole term}} + \cdots \end{aligned} \tag{21}$$

in the cylindrically symmetric case about the z-axis.

Average E - Field Inside a Sphere Containing an Arbitrary Charge Distribution

Consider first determining the average electric field intensity inside a sphere of radius R containing a point charge Q situated a distance r' from the center O of the sphere with the z-axis taken along the line passing through the point charge Q and the origin O.



Average E - Field Inside a Sphere Containing an Arbitrary Charge Distribution

By symmetry, the average **E**-field over the spherical volume must be along the z-axis. The average E - field over the spherical region is then given by

$$\overline{E_z} = \frac{1}{\tau} \iiint_{\tau} E_z d^3 r \qquad (22)$$

where $\tau = \frac{4}{3}\pi R^3$ denotes the volume of the sphere.

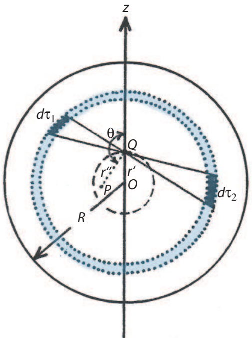
This integral can be separated into two parts, one over the spherical shell τ_1 between r' and R, and one over the sphere τ_0 of radius r':

$$\overline{E_z} = \frac{1}{\tau} \iiint_{\tau_0} E_z d^3 r + \underbrace{\frac{1}{\tau} \iiint_{\tau_1} E_z d^3 r}_{0} . \qquad (23)$$

Average E - Field Inside a Sphere Containing an Arbitrary Charge Distribution

The integral over the spherical shell τ_1 vanishes because:

- $d\Omega$ intercepts volume elements $d\tau_1$ & $d\tau_2$ in each spherical shell.
- E_z decreases as the square of the distance from Q whereas $d\tau$ increases as the square of this distance. $\therefore$ product is constant.
- E_z is positive at $d\tau_1$ while it is negative at $d\tau_2$, the two contributions to the integral then canceling.



Average E - Field Inside a Sphere Containing an Arbitrary Charge Distribution

The remaining integral over the inner sphere τ_0 may then be evaluated using spherical polar coordinates with origin at the position of the point charge Q. At the point P,

$$E_z = \mathbf{E} \cdot \hat{\mathbf{1}}_z = \frac{Q}{4\pi\epsilon_0 r''^2} \hat{\mathbf{1}}_{r''} \cdot \hat{\mathbf{1}}_z = \frac{Q}{4\pi\epsilon_0 r''^2} \cos\theta,$$

so that

$$\begin{aligned}
\iiint_{\tau_0} E_z d^3 r &= \int_{\phi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \int_{r''=0}^{-2r'\cos\theta} \frac{Q\cos\theta}{4\pi\epsilon_0 r''^2} r''^2 \sin\theta dr'' d\theta d\phi \\
&= \frac{Q}{4\pi\epsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_{\pi/2}^{\pi} \sin\theta\cos\theta \left[\underbrace{\int_0^{-2r'\cos\theta} dr''}_{-2r'\cos\theta} \right] d\theta \\
&= -\frac{Qr'}{\epsilon_0} \int_{\pi/2}^{\pi} \cos^2\theta \sin\theta d\theta = -\frac{Qr'}{3\epsilon_0}.
\end{aligned}$$

Average E - Field Inside a Sphere Containing an Arbitrary Charge Distribution

Hence

$$\overline{E_z} = \frac{1}{\frac{4}{3}\pi R^3}\left(-\frac{Qr'}{3\epsilon_0}\right) = -\frac{Qr'}{4\pi\epsilon_0 R^3}. \tag{24}$$

Because Qr' is the dipole moment of the charge Q relative to the origin O of the sphere, then

$$\boxed{\overline{\mathbf{E}} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}} \tag{25}$$

For an arbitrary charge distribution $\varrho(\mathbf{r})$ inside the sphere, the electrostatic field is the superposition of the fields due to the individual charge elements $\varrho(\mathbf{r})d^3r$. The average E - field within the sphere is then given by Eq. (25) with $\mathbf{p}$ given by the dipole moment of the arbitrary charge distribution within the sphere of radius R, as given by Eq. (8).